Propositional Resolution

Valentin Goranko

School of Mathematics, University of the Witwatersrand
Johannesburg, South Africa

February 2008

The Propositional Resolution rule

$$\frac{A \vee C,\ B \vee \neg C}{A \vee B}$$

The formula $A \vee B$ is called a resolvent of $A \vee C$ and $B \vee \neg C$, denoted $Res(A \vee C, B \vee \neg C)$.

EXERCISE: Show that the Resolution rule is logically valid.

Consequently, it preserves satisfiability of the clause set.

Clausal normal forms

- A clause is essentially an elementary disjunction $l_1 \vee \ldots \vee l_n$, but written as a set of literals $\{l_1, \ldots, l_n\}$.
- The empty clause $\{\}$ is a clause containing no literals.
- A unit clause is a clause containing only one literal.
- A clausal form is a (possibly empty) set of clauses, written as a list: $C_1 \ldots C_k$. It represents the *conjunction* of these clauses.

Thus, every CNF can be re-written in a clausal form, and therefore every propositional formula is equivalent to one in a clausal form.

EXAMPLE: the clausal form of the CNF-formula $(p \vee \neg q \vee \neg r) \wedge \neg p \wedge (\neg q \vee r)$ is $\{p, \neg q, \neg r\}\{\neg p\}\{\neg q, r\}$.

Note that the empty clause $\{\}$ is not satisfiable (being an empty disjunction), while the empty set of clauses $\emptyset$ is satisfied by any truth assignment (being an empty conjunction).

Clausal Propositional Resolution rule

The Propositional Resolution rule can be rewritten for clauses:

$$\frac{\{A_1, \ldots, C, \ldots, A_m\} \ \{B_1, \ldots, \neg C, \ldots, B_n\}}{\{A_1, \ldots, A_m, B_1, \ldots, B_n\}}.$$

The clause $\{A_1, \ldots, A_m, B_1, \ldots, B_n\}$ is called a resolvent of the clauses $\{A_1, \ldots, C, \ldots, A_m\}$ and $\{B_1, \ldots, \neg C, \ldots, B_n\}$.

Example

$$\frac{\{p, q, \neg r\} \ \{\neg q, \neg r\}}{\{p, \neg r\}},$$

$$\frac{\{\neg p, q, \neg r\} \ \{r\}}{\{\neg p, q\}},$$

$$\frac{\{\neg p\} \ \{p\}}{\{\}}.$$

Some remarks

Note that two clauses can have more than one resolvent, e.g.:

$$\frac{\{p, \neg q\}\{\neg p, q\}}{\{p, \neg p\}}, \quad \frac{\{p, \neg q\}\{\neg p, q\}}{\{\neg q, q\}}.$$

However, it is wrong to apply the Propositional Resolution rule for both pairs of complementary literals simultaneously and obtain

$$\frac{\{p, \neg q\}\{\neg p, q\}}{\{\}}.$$

Sometimes, the resolvent can (and should) be simplified, by removing duplicated literals:

$$\{A_1, \ldots, C, C, \ldots, A_m\} \;\Rightarrow\; \{A_1, \ldots, C, \ldots, A_m\}.$$

For instance:

$$\frac{\{p, \neg q, \neg r\}\{q, \neg r\}}{\{p, \neg r\}}$$

Propositional resolution as a deductive system

The underlying idea of *Propositional Resolution* is like the one of Semantic Tableau: in order to prove the validity of a logical consequence $A_1, \ldots, A_n \models B$, show that there is no truth assignment which falsifies it, i.e., show that the formulae $A_1, \ldots, A_n$ and $\neg B$ *cannot be satisfied simultaneously*.

That is done by transforming the formulae $A_1, \ldots, A_n$ and $\neg B$ to a clausal form, and then using repeatedly the Propositional Resolution rule in attempt to derive the empty clause $\{\}$.

Since $\{\}$ is not satisfiable, its derivation means that $A_1, \ldots, A_n$ and $\neg B$ cannot be satisfied together.
Then, the logical consequence $A_1, \ldots, A_n \models B$ holds.

Alternatively, after finitely many applications of the Propositional Resolution rule, no new applications of the rule remain possible. If the empty clause is not derived by then, it cannot be derived at all, and hence the $A_1, \ldots, A_n$ and $\neg B$ can be satisfied together, so the logical consequence $A_1, \ldots, A_n \models B$ does not hold.

Propositional resolution derivation: Example 1

Prove $p \rightarrow q, q \rightarrow r, \models p \rightarrow r$.

First, transform $p \rightarrow q, q \rightarrow r, \neg(p \rightarrow r)$ to clausal form:

$$C_1 = \{\neg p, q\}, C_2 = \{\neg q, r\}, C_3 = \{p\}, C_4 = \{\neg r\}.$$

Now, applying Propositional Resolution successively:

$C_5 = Res(C_1, C_3) = \{q\}$;

$C_6 = Res(C_2, C_5) = \{r\}$;

$C_7 = Res(C_4, C_6) = \{\}$.

The derivation of the empty clause completes the proof.

Propositional resolution derivation: Example 2

Check if $(\neg p \rightarrow q), \neg r \models p \vee (\neg q \wedge \neg r)$.

First, transform $(\neg p \rightarrow q), \neg r, \neg(p \vee (\neg q \wedge \neg r))$ to clausal form:

$$C_1 = \{p, q\}, C_2 = \{\neg r\}, C_3 = \{\neg p\}, C_4 = \{q, r\}.$$

Now, applying Propositional Resolution successively:

$C_5 = Res(C_1, C_3) = \{q\}$;

$C_6 = Res(C_2, C_4) = \{q\}$;

At this stage, no new applications of the Propositional Resolution rule are possible, hence the empty clause is not derivable.

Therefore, $(\neg p \rightarrow q), \neg r \not\models p \vee (\neg q \wedge \neg r)$.