Deductive systems. Soundness and completeness

Mathematical Logic Honours 2008

Valentin Goranko

1 The idea of deductive systems

A formal system for logical inference is called a **deductive system**. The purpose of a deductive system is to capture the notion of *logical consequence* by simulating it with *deductive consequence*. A prototype of a deductive system can be found in Euclid's *Elements* which present a systematic development of elementary geometry, based on several simple assumptions about points and lines (such as "Every two different points determine exactly one line", "For every line there is a point not belonging to that line", etc.). Using these and (informal) logical reasoning, other geometric facts are derived, and thus the entire body of Euclidean geometry is eventually built. The concept of a *formal* deductive system gradually emerged much later, notably in the ideas of Leibniz for a *characteristica universalis* (universal language) and a *calculus ratiocinator* (calculus of reasoning), and was explicitly formulated in the early 20th century by Hilbert, who used it to revise and reconstruct Euclid's work on foundations of geometry. The first prototypes of formal deductive systems were constructed in the end of the 19th century by Peano (Peano arithmetic) and Frege ('Begriffsschrift'), and further developed by Russell and Whitehead ('Principia Mathematica'), and Hilbert and Ackermann ('Principles of Mathematical Logic').

There are different types of deductive systems, but they all share several common principles:

- A deductive system works with formal expressions called **formulae** which are words (finite strings of symbols) in a **formal language** and have a precise **syntax**. (In this aspect a deductive system is like a programming language.) In this section we discuss deductive systems working on propositional formulae.

- A deductive system is based on a set of specified **rules of inference**, like those studied in the previous section. By applying systematically these rules, we **derive (infer, deduce)** formulae from a specified set of formulae called **assumptions (premisses)**, possibly also using other, already proven formulae. Some deductive systems also allow for an initial set of formulae, called **axioms**, to be accepted as proven without applying any rules of inference. Thus, the axioms can always be used as premises in the derivations.

  A formula, derived in a deductive system $\mathcal{D}$ from no assumptions (apart from axioms), is called a **theorem** of $\mathcal{D}$.

- A very important aspect of deductive systems is that the derivations are *completely mechanical procedures*, which do not require any intelligence or even understanding of the meaning of the formulae or rules involved; in fact such meaning need not be specified at all. Thus, derivations in a deductive system can be performed by a mechanical device, such as a computer, without any human intervention, as long as the axioms and rules of inference of the deductive system have been programmed into it.

While deductive systems are not explicitly concerned with the meaning *(semantics)* of the formulae they derive, they are designed with the purpose to derive *valid, and only valid* logical consequences from the assumptions. A deductive system with this property is called **sound**, or **(logically) correct**. In particular, every theorem of such a deductive system must always be true, i.e. in the case of propositional logic it must be a tautology.

If a deductive system is powerful enough to derive *every* valid logical consequence, it is called **complete**. In particular, a complete deductive system for propositional logic can derive *every* tautology.

Note that the soundness of a deductive system can be guaranteed, and proved, quite easily, as long as the following two conditions hold:

(i) *All axioms (if any) must be true.*

(ii) *All rules of inference must be valid*, i.e. they must always produce true conclusions when applied to true assumptions.

Thus, the truth propagates from the axioms to all theorems.

2 Soundness and completeness of a deductive system

Here we make the notions of soundness and completeness of a deductive system precise.

Let $\mathbf{D}$ be any deductive system. If a formula A can be derived from a set of assumptions Γ in $\mathbf{D}$, we denote this by $\Gamma \vdash_{\mathbf{D}} A$. In particular, if $\Gamma = \emptyset$ we write $\vdash_{\mathbf{D}} A$. We often refer to Γ as a *theory*.

DEFINITION (SOUNDNESS1). The system $\mathbf{D}$ is **sound** if for every theory Γ and a formula A:

$$\Gamma \vdash_{\mathbf{D}} A \text{ implies } \Gamma \models A.$$

DEFINITION (DEDUCTIVE CONSISTENCY). A theory Γ is **consistent in D** (or, just **D -consistent)** if there is no formula A such that $\Gamma \vdash_{\mathbf{D}} A$ and $\Gamma \vdash_{\mathbf{D}} \neg A$. Otherwise, Γ is **D-inconsistent**.

DEFINITION (SOUNDNESS2). A deductive system $\mathbf{D}$ is **sound** if for every theory Γ,

$$\text{if } \Gamma \text{ is satisfiable then } \Gamma \text{ is } \mathbf{D}\text{-consistent.}$$

Exercise 2.1 *Prove that the two definitions of soundness are equivalent.*

DEFINITION (COMPLETENESS1). A deductive system $\mathbf{D}$ is **complete** if for every theory Γ and a formula A,

$$\Gamma \models A \text{ implies } \Gamma \vdash_{\mathbf{D}} A.$$

DEFINITION (COMPLETENESS2). A deductive system $\mathbf{D}$ is **complete** if for every theory Γ,

$$\text{if } \Gamma \text{ is } \mathbf{D}\text{-consistent then } \Gamma \text{ is satisfiable.}$$

Exercise 2.2 *Prove that the two definitions of completeness are equivalent.*