Classical deductive systems

Axiomatic Systems

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Hilbert-style axiomatic systems

- Based on axioms (or, axiom schemes), and only one or two simple rules of inference.
- Relatively easy to extract from the semantics and reason about.

  In particular, suitable to do induction on derivations.
- Practically not very convenient and useful, because the derivations are not well-structured.
- In particular, not suitable for automated reasoning.

The axiomatic system H for the classical propositional logic

Axioms schemes for $\neg$ and $\rightarrow$:

$(\rightarrow 1)$ $A \rightarrow (B \rightarrow A)$;

$(\rightarrow 2)$ $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$;

$(\rightarrow 3)$ $(\neg B \rightarrow \neg A) \rightarrow ((\neg B \rightarrow A) \rightarrow B)$.

The only rule of inference: **Modus ponens:**

$$\frac{A, A \rightarrow B}{B}.$$

Axioms schemes for $\wedge$:

$(\wedge 1)$ $(A \wedge B) \rightarrow A$;

$(\wedge 2)$ $(A \wedge B) \rightarrow B$;

$(\wedge 3)$ $(A \rightarrow B) \rightarrow ((A \rightarrow C) \rightarrow (A \rightarrow B \wedge C))$.

Axioms schemes for $\vee$:

$(\vee 1)$ $A \rightarrow A \vee B$;

$(\vee 2)$ $B \rightarrow A \vee B$;

$(\vee 3)$ $(A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow (A \vee B \rightarrow C))$.

Derivations in $\mathbf{H}$

Formula derivable from a set of assumptions: $\Gamma \vdash_{\mathbf{H}} A$

if there is a finite sequence of formulae $A_1, ..., A_n$, such that for every $i \leq n$:

A_i is either an instance of an axiom of $\mathbf{H}$,

or a formula from Γ,

or is obtained from some A_j, A_k for $j, k < i$, by applying the rule Modus Ponens.

A is a **theorem of $\mathbf{H}$**, if $\varnothing \vdash_{\mathbf{H}} A$, also denoted $\vdash_{\mathbf{H}} A$.

Example: $\vdash_{\mathbf{H}} (p \wedge (p \rightarrow q)) \rightarrow q$:

1. $\vdash_{\mathbf{H}} (p \wedge (p \rightarrow q)) \rightarrow p$, by Axiom $(\wedge 1)$;
2. $\vdash_{\mathbf{H}} (p \wedge (p \rightarrow q)) \rightarrow (p \rightarrow q)$, by Axiom $(\wedge 2)$;
3. $\vdash_{\mathbf{H}} ((p \wedge (p \rightarrow q)) \rightarrow (p \rightarrow q)) \rightarrow (((p \wedge (p \rightarrow q)) \rightarrow p) \rightarrow ((p \wedge (p \rightarrow q)) \rightarrow q))$, by Axiom $(\rightarrow 2)$;
4. $\vdash_{\mathbf{H}} (p \wedge (p \rightarrow q)) \rightarrow p) \rightarrow ((p \wedge (p \rightarrow q)) \rightarrow q)$, by 2,3 and Modus Ponens;
5. $\vdash_{\mathbf{H}} (p \wedge (p \rightarrow q)) \rightarrow q$, by 1,4 and Modus Ponens.

Using the Deduction Theorem

Deduction Theorem: For any set of formulae Γ and formulae A and B:

$$\Gamma \cup \{A\} \vdash_{\mathbf{H}} B \text{ iff } \Gamma \vdash_{\mathbf{H}} A \to B.$$

Example $\vdash_{\mathbf{H}} (p \wedge (p \to q)) \to q$ revisited:

1. $p \wedge (p \to q) \vdash_{\mathbf{H}} p$, by Axiom $(\wedge 1)$ and the Deduction Theorem;
2. $p \wedge (p \to q) \vdash_{\mathbf{H}} p \to q$ by Axiom $(\wedge 2)$ and the Deduction Theorem;
3. $p \wedge (p \to q) \vdash_{\mathbf{H}} q$ by 1,2, and Modus Ponens;
4. $\vdash_{\mathbf{H}} (p \wedge (p \to q)) \to q$ by 3 and the Deduction Theorem.

Example: $p, \neg p \vdash_{\mathbf{H}} q$:

1. $\neg p \vdash_{\mathbf{H}} \neg q \to \neg p$, by Axiom $(\to 1)$ and the Deduction Theorem;
2. $\vdash_{\mathbf{H}} (\neg q \to \neg p) \to (p \to q)$, by Axiom $(\to 3)$;
3. $\neg p \vdash_{\mathbf{H}} p \to q$, by 1,2, and Modus Ponens;
4. $p, \neg p \vdash_{\mathbf{H}} q$, by 3 and the Deduction Theorem.

Extending H to first-order logic

Axioms for the system **QH** (with $\forall$ as the only quantifier in the language):

(Ax$\forall$1) $\forall x(A(x) \rightarrow B(x)) \rightarrow (\forall x A(x) \rightarrow \forall x B(x))$;

(Ax$\forall$2) $\forall x A(x) \rightarrow A(t/x)$ where t is any term free for substitution for x in A;

(Ax$\forall$3) $A \rightarrow \forall x A$ where x is not free in the formula A;

The rule of *generalization:*

$$\frac{A}{\forall x A}.$$

Axioms for the equality:

(Ax = 1) $x = x$

(Ax = 2) $x = y \rightarrow y = x$

(Ax = 3) $x = y \wedge y = z \rightarrow x = z$.

(Ax f) $x_1 = y_1 \wedge ... \wedge x_n = y_n \rightarrow f(x_1, ..., x_n) = f(y_1, ..., y_n)$ for every n-ary functional symbol p.

(Ax r) $x_1 = y_1 \wedge ... \wedge x_n = y_n \rightarrow (p(x_1, ..., x_n) \rightarrow p(y_1, ..., y_n))$ for every n-ary predicate symbol p.